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Improved Approximations for Wave Structure Functions in a Turbulent Atmosphere

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IMPROVED APPROXIMATIONS FOR WAVE STRUCTURE FUNCTIONS IN A TURBULENT ATMOSPHERE

1. INTRODUCTION

A basic knowledge of the wave structure function and its use in evaluating the effects of turbulence on a light wave propagating through the atmosphere is assumed. The interested reader is referred to review articles [1,2] or books [3,4] on this topic. I show below that the familiar form of the plane wave structure function [5], $D(r) = 6.88(r/r_0)^{5/3}$, is a poor approximation unless $r/L_O < 0.01$, where L_O is the outer scale length. Similar remarks apply to the spherical wave structure function. Since r is often on the order of 1 m, while for many problems L_O is in the range of 10 to 100 m (cf. Beland [1], p. 168, Ishimaru [3], p. 360, or Winker [6]), better approximations are clearly desirable. This report presents approximations that are good for values of r/L_O up to a few tenths. This work extends that of Lutomirski and Yura [7] for a pure Kolmogorov spectral density and of Andrews et al. [8] for its von Karman modification by including a higher-order term in the expansion and by showing explicitly the range over which the approximations are accurate. Failing to state this range is the primary shortcoming of the literature.

I use the standard descriptors of turbulence: L_I and L_O are the inner and outer scale lengths, $K_I = 2\pi/L_I$ and $K_O = 2\pi/L_O$ are their corresponding wave numbers in Fourier space, $\Phi_n(K) = 0.033C_n^2/K^{11/3}$, $K_O \leq K \leq K_I$, is the Kolmogorov power spectral density of the turbulence-induced fluctuations in the index of refraction of the atmosphere, and C_n^2 is the conventional structure constant.

The region of interest for this report is $r > L_I$. The fact that $D(r)$ has a different form [1-4] for $r < L_I$ will not be considered, because, for the problems of interest here, r_0 is at least a few centimeters, while L_I is generally taken to be a few millimeters [2-4], which means that $D(r) \approx 0$ when $r \leq L_I$. Also for this reason, neither the Tartarski [9] nor the Hill [4,10] corrections to the Kolmogorov power spectrum will be considered: they apply to high values of K , hence they have an appreciable effect on the structure function only for $r < L_I$. Consequently, the approximation $r \gg L_I$ will be used, which means that L_I is taken as zero ($K_I = \infty$) in the integrals that give the structure function.

2. THE PLANE WAVE STRUCTURE FUNCTION

2.1 The Kolmogorov Power Spectral Density of Turbulence

I begin with the plane wave structure function from, for example, Eq. (8.6-29) of Goodman [9], which applies to a path of length Z through a region of turbulence described by constant C_n^2 and includes the approximations of setting the outer scale to infinity and the inner scale to zero. To show the effect of a finite outer scale, I take the lower limit of the integral as K_O instead of 0, change the variable of integration to $u = Kr$, and put the result in a form that will be useful below:

$$\begin{aligned}
D(r) &= 8\pi^2 k^2 Z \times 0.033 C_n^2 \int_{K_o}^{\infty} \frac{1 - J_0(Kr)}{K^{11/3}} K dK \\
&= \frac{6(2\pi)^2 0.033 C_n^2 k^2 Z}{5 K_o^{5/3}} \left[\frac{5}{3} (K_o r)^{5/3} \int_{K_o r}^{\infty} \frac{1 - J_0(u)}{u^{8/3}} du \right], \tag{1}
\end{aligned}$$

where $k = 2\pi/\lambda$. Equation (1) expresses the structure function as a multiplicative factor times the quantity in brackets, which is a function only of the ratio r/L_o . Following Andrews et al. [8], I call this quantity the normalized structure function and put it in a form that will be useful below:

$$\begin{aligned}
D^* \left(\frac{r}{L_o} \right) &\equiv \frac{5}{3} (K_o r)^{5/3} \int_{K_o r}^{\infty} \frac{1 - J_0(u)}{u^{8/3}} du \\
&= \frac{5}{3} (K_o r)^{5/3} \left\{ \int_0^{\infty} \frac{1 - J_0(u)}{u^{8/3}} du - \int_0^{K_o r} \frac{1 - J_0(u)}{u^{8/3}} du \right\}. \tag{2}
\end{aligned}$$

$D^*(r/L_o)$ is evaluated by numerical integration of the integral in the first line of Eq. (2) and plotted in Figs. 1 and 2, along with the analytic approximations described below. D^* is normalized to an asymptotic value of 1, a fact that is most easily demonstrated analytically by changing the variable of integration in the first line of Eq. (2) to $v = u/K_o r$. The result is

$$D^* \left(\frac{r}{L_o} \right) \equiv \frac{5}{3} \int_1^{\infty} \frac{1 - J_0(K_o r v)}{v^{8/3}} dv \rightarrow \frac{5}{3} \int_1^{\infty} \frac{dv}{v^{8/3}} = 1 \quad \text{as} \quad \frac{r}{L_o} \rightarrow \infty, \tag{3}$$

since $J_0(x) \rightarrow 0$ as $x \rightarrow \infty$.

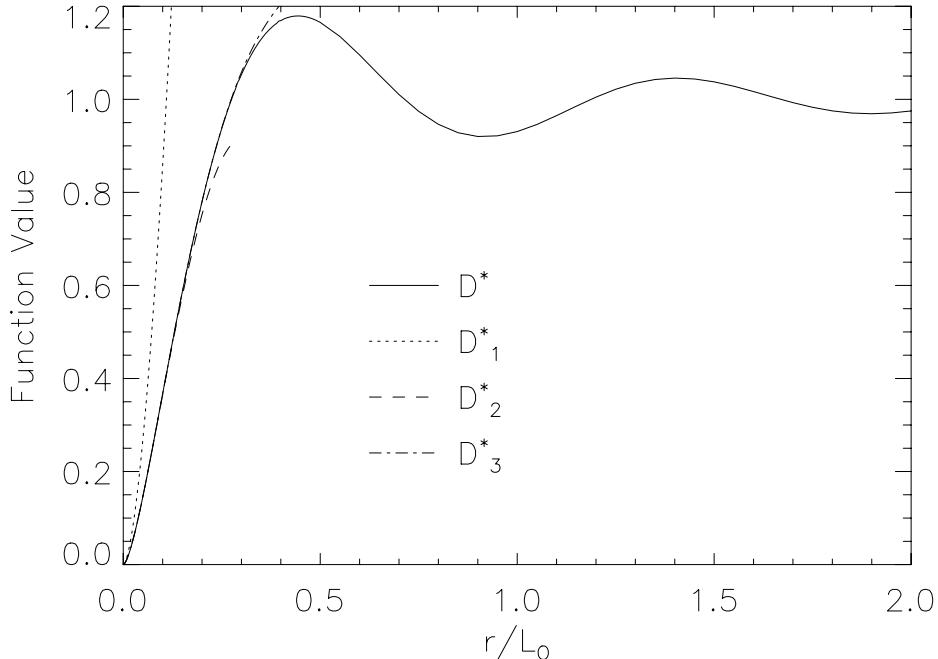


Fig. 1 — The four functions plotted are taken from Eqs. (2), (5), (6), and (7) for a plane wave and a Kolmogorov spectral density. The solid line is the numerically calculated wave structure function; the other lines are analytic approximations (see text).

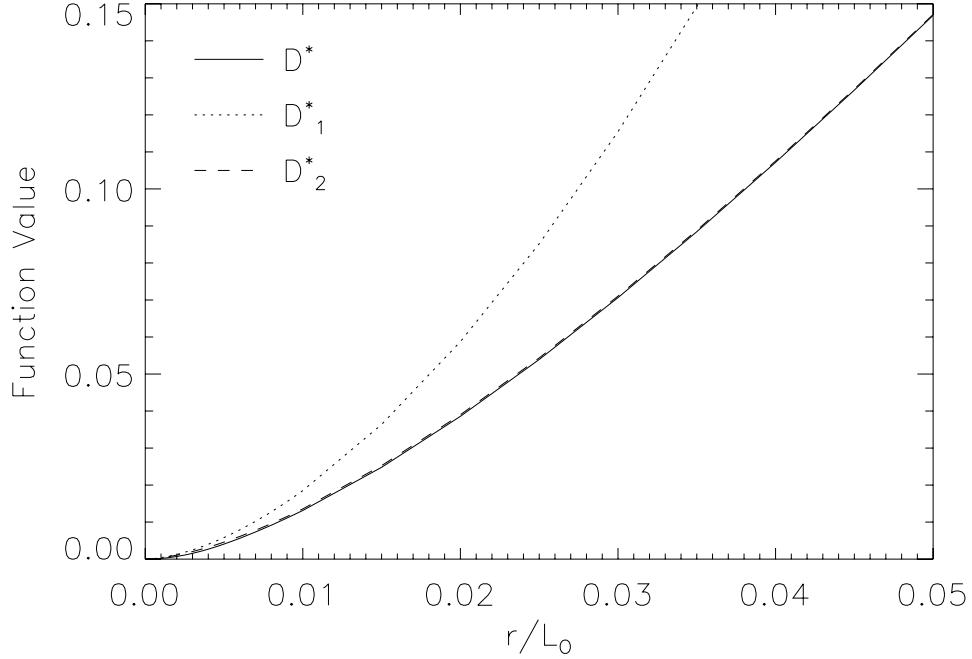


Fig. 2 — Detail of Fig. 1, with range limited to $r/L_O \leq 0.05$ and D^{*3} not plotted. D^* and D^{*2} are nearly identical in this region. From Eq. (5), D^{*1} is the normalized form of the usual approximation to the wave structure function, that is, $D^{*1}(r/L_O) \propto (r/L_O)^{5/3}$.

For most problems of interest, $r \ll L_O$. In order to find analytic expressions for D^* in this regime, I take the first integral in the second line of Eq. (2) from Eq. (8.6-30) of Goodman [9] and the second from Eq. (A1) of this report's Appendix. The result is a power-series expansion, with powers 0, 1/3, and 7/3 of the quantity in braces in Eq. (2):

$$\begin{aligned}
 D^*\left(\frac{r}{L_O}\right) &\approx \frac{5}{3}(K_O r)^{5/3} \left[\frac{\pi}{2^{5/3} \Gamma^2(11/6)} - \frac{3}{4}(K_O r)^{1/3} + \frac{3}{7 \times 64}(K_O r)^{7/3} \right] \\
 &= 35.66 \left(\frac{r}{L_O}\right)^{5/3} \left[1.118 - 1.384 \left(\frac{r}{L_O}\right)^{1/3} + 0.488 \left(\frac{r}{L_O}\right)^{7/3} \right] \\
 &= 39.86 \left(\frac{r}{L_O}\right)^{5/3} - 49.35 \left(\frac{r}{L_O}\right)^2 + 17.40 \left(\frac{r}{L_O}\right)^4.
 \end{aligned} \tag{4}$$

I denote by the subscripts 1, 2, and 3, the one-, two-, and three-term expansions of D^* and of D . Thus,

$$D_1^*\left(\frac{r}{L_O}\right) = 39.86 \left(\frac{r}{L_O}\right)^{5/3}, \tag{5}$$

$$D_2^*\left(\frac{r}{L_O}\right) = 39.86 \left(\frac{r}{L_O}\right)^{5/3} - 49.35 \left(\frac{r}{L_O}\right)^2, \tag{6}$$

and

$$D_3^* \left(\frac{r}{L_o} \right) = 39.86 \left(\frac{r}{L_o} \right)^{5/3} - 49.35 \left(\frac{r}{L_o} \right)^2 + 17.4 \left(\frac{r}{L_o} \right)^4. \quad (7)$$

These functions are plotted in Figs. 1 and 2, from which we see that D^*_1 is reasonably accurate only for $r/L_o < 0.01$, while D^*_2 extends this range dramatically to $r/L_o \leq 0.2$, and D^*_3 yields a further improvement to $r/L_o \leq 0.4$. Higher-order terms can be calculated by continuing the expansion of $1 - J_0(u)$ in Eq. (2), but are unlikely to be useful.

Substitution of Eq. (4) into Eq. (1) yields

$$\begin{aligned} D(r) &\approx \frac{6(2\pi)^2 0.033 C_n^2 k^2 Z}{5 K_o^{5/3}} \frac{5}{3} (K_o r)^{5/3} \left[1.118 - 1.384 \left(\frac{r}{L_o} \right)^{1/3} + 0.488 \left(\frac{r}{L_o} \right)^{7/3} \right] \\ &= 2.91 C_n^2 k^2 Z r^{5/3} \left[1 - 1.238 \left(\frac{r}{L_o} \right)^{1/3} + 0.436 \left(\frac{r}{L_o} \right)^{7/3} \right] \\ &= 6.88 \left(\frac{r}{r_0} \right)^{5/3} \left[1 - 1.238 \left(\frac{r}{L_o} \right)^{1/3} + 0.436 \left(\frac{r}{L_o} \right)^{7/3} \right], \end{aligned} \quad (8)$$

where the definition of the Fried coherence length [5], $r_0 \equiv [6.88/(2.91 C_n^2 k^2 Z)]^{3/5}$, has been used. $D_1(r)$ is the structure function for infinite outer scale,

$$D_1(r) = 6.88 \left(\frac{r}{r_0} \right)^{5/3}, \quad (9)$$

and is the form most often used, with the qualification that it is valid only when $r \ll L_o$, a qualification that Fig. 2 shows to be very true: even when r is only 1% of L_o , $D_1(r)$ exceeds $D(r)$ by about one-third (35%, to be exact). Probably the most useful approximation is

$$D_2(r) = 6.88 \left(\frac{r}{r_0} \right)^{5/3} \left[1 - 1.24 \left(\frac{r}{L_o} \right)^{1/3} \right], \quad (10)$$

which is good for $r/L_o \leq 0.2$ and is identical with Eq. (6) of Lutomirski and Yura [7], who did the calculation the same way, but kept only the first term in the expansion of $1 - J_0(u)$. $D_3(r)$ is given by the last line of Eq. (8).

2.2 The Von Karman Power Spectral Density of Turbulence

The von Karman spectral density is the Kolmogorov spectral density with a roll-off applied at low frequencies: $\Phi_n(K) = 0.033 C_n^2 / (K^2 + K_o^2)^{11/6}$, as shown in Eq. (8.4-16) of Goodman [9] (the exponential high-frequency attenuation also shown in this equation is the Tartarski correction to the Kolmogorov spectrum and, as explained in the introduction, is not used here). This allows the lower limit of the integral in Eq. (1) to be zero, as done by, for two examples, Andrews et al. [8] and Goodman [9], which corresponds to the physical assumption that there is no low-frequency cut-off in the turbulence power spectrum. (The reader's attention is called to the notational oddity of using the symbol K_o for the von Karman roll-off frequency, then extending the lower limit of the integral to zero, which, under the definition of K_o used for the Kolmogorov density, would imply $K_o = 0$.) The assumption of no low-

frequency cut-off may be poor, especially in near-surface propagation problems, and in any case there is no obvious connection between the cut-off frequency and the roll-off frequency. I therefore use the parameter a to relate the cut-off frequency, aK_o , to the roll-off frequency, K_o , and assume $0 \leq a \leq 1$. When this form is inserted into Goodman's Eq. (8.6-29), Eq. (1) becomes

$$\begin{aligned} D_{vK}(r) &= 8\pi^2 0.033 C_n^2 k^2 Z \int_{aK_o}^{\infty} \frac{1 - J_0(Kr)}{(K^2 + K_o^2)^{11/6}} K dK \\ &= \frac{6(2\pi)^2 0.033 C_n^2 k^2 Z}{5K_o^{5/3}} \left[\frac{5}{3} (K_o r)^{5/3} \int_{aK_o r}^{\infty} \frac{1 - J_0(u)}{(u^2 + K_o^2 r^2)^{11/6}} u du \right], \end{aligned} \quad (11)$$

so the normalized structure function is, as in Eq. (2),

$$D_{vK}^* \left(\frac{r}{L_o} \right) = \frac{5}{3} (K_o r)^{5/3} \left\{ \int_0^{\infty} \frac{1 - J_0(u)}{(u^2 + K_o^2 r^2)^{11/6}} u du - \int_0^{aK_o r} \frac{1 - J_0(u)}{(u^2 + K_o^2 r^2)^{11/6}} u du \right\}. \quad (12)$$

$D_{vK}^*(r/L_o)$ is evaluated by numerical integration of Eq. (12) and plotted in Figs. 3 and 4, along with the analytic approximations described below.

The quantity in braces in Eq. (12) is less analytically tractable than is the similar expression in Eq. (2). For $a = 0$, the second integral disappears and, as shown by Andrews et al. [8] (by using hypergeometric functions!), the first can again, as in Eq. (4), be evaluated as a power series in $K_o r$. Taking $K_o r = 0$, comparison of Eqs. (12), (2), and (4) shows that the coefficient of order 0 is again 1.118. The coefficient of order 1/3 was found by Andrews et al. to be -1.485 . As in Eq. (4), these values are multiplied by 35.66 to yield 39.86 and -59.2 , respectively, in Eq. (13). The second integral in Eq. (12) is given by Eq. (A2) in the Appendix and shown to change the second coefficient from -1.485 to -1.511 (a 1.8% change) when $a = 1$, less when $a < 1$. I consider this difference to be negligible and only the value -59.2 is used in Eq. (13) and in Figs. 3 and 4, that is, the second term in Eq. (13) is approximated as being independent of a . Figure 3 shows the validity of this approximation: D_{vK}^* is essentially independent of a over the region in which $D_{vK,2}$ is good. The third coefficient in Eq. (13) was found by the simple expedient of adopting the form of Eq. (4), replacing -49.35 by -59.2 , and varying the last coefficient until a visually good fit was obtained. The astute reader will notice that this coefficient was chosen to extend the region over which $D_{vK,3}$ gives a good approximation; it is not the coefficient that would be obtained in a Taylor series expansion (the error of the latter would probably be monotonic, as in Fig. 1, not first negative, then positive). The result is

$$D_{vK,3}^* \left(\frac{r}{L_o} \right) \approx 39.86 \left(\frac{r}{L_o} \right)^{5/3} - 59.2 \left(\frac{r}{L_o} \right)^2 + \begin{cases} 120 & \text{for } a = 0 \\ 80 & \text{for } a = 1 \end{cases} \left(\frac{r}{L_o} \right)^4, \quad (13)$$

which is reasonably valid for $r/L_o \leq 0.25$. $D_{vK,1}$ and $D_{vK,2}$ are also given by Eq. (13). The von Karman equivalent of Eq. (8) is

$$D_{vK,3}(r) \approx 6.88 \left(\frac{r}{r_0} \right)^{5/3} \left[1 - 1.485 \left(\frac{r}{L_o} \right)^{1/3} + \begin{cases} 3 & \text{for } a = 0 \\ 2 & \text{for } a = 1 \end{cases} \left(\frac{r}{L_o} \right)^{7/3} \right]. \quad (14)$$

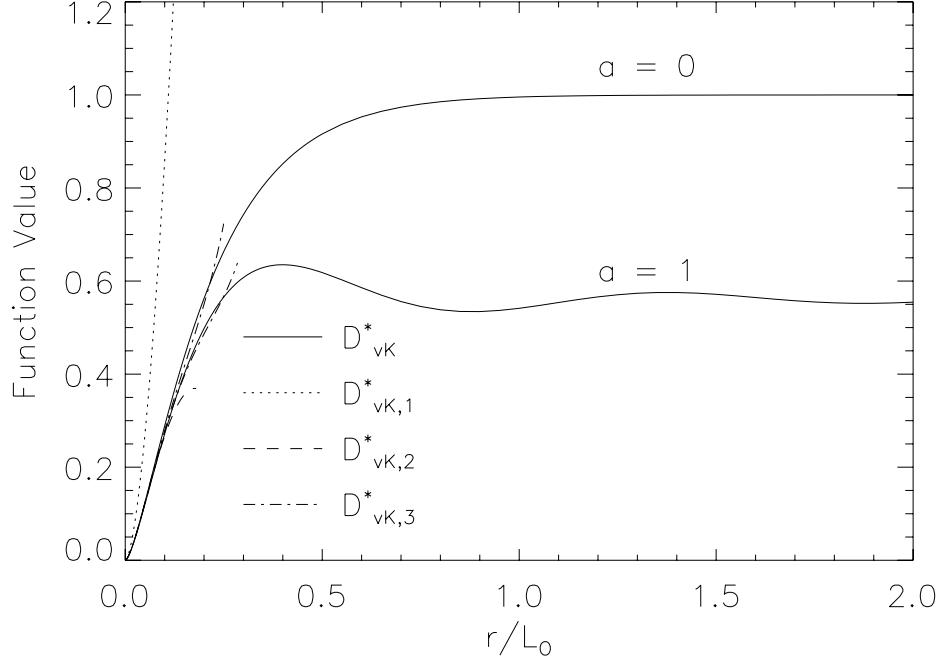


Fig. 3 — Similar to Fig. 1, but with von Karman spectral density and functions given by Eqs. (12) and (13). $D^*_{vK,1} = D^*_1$ is the same as in Figs. 1 and 2. For $a = 1$ the low-frequency cut-off in the turbulence power spectrum is the same as the von Karman roll-off frequency, for $a = 0$ the spectrum extends to zero frequency. Asymptotic values are given in Eq. (15).

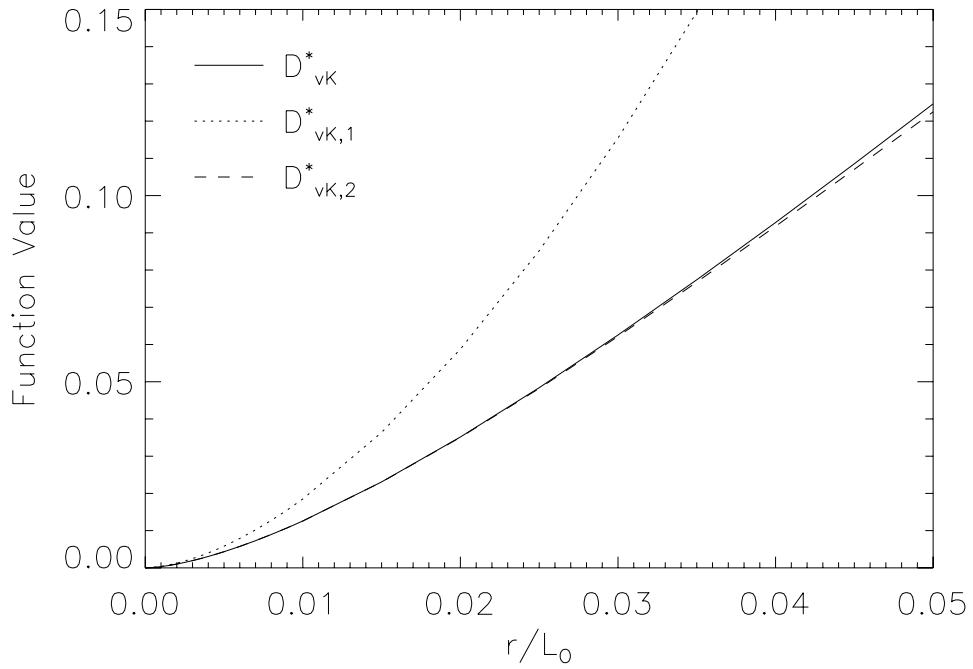


Fig. 4 — Detail of Fig. 3. D^*_{vK} is plotted for $a = 0$. For $a = 1$, D^*_{vK} changes by less than 3% in the region shown.

Omitting the last term inside the brackets in Eq. (14) yields $D_{vK,2}$, which is the same as Eq. (5) of Andrews et al. for $L_I \ll r \ll L_O$.

The asymptotic value in Fig. 3 is calculated as in Eq. (3):

$$\begin{aligned} D_{vK}^* \left(\frac{r}{L_O} \right) &= \frac{5}{3} \int_a^\infty \frac{1 - J_0(K_O r v)}{(v^2 + 1)^{11/6}} v dv \rightarrow \frac{5}{6} \int_a^\infty \frac{d(v^2)}{(v^2 + 1)^{11/6}} \text{ as } \frac{r}{L_O} \rightarrow \infty \\ &= \frac{5}{6} \left[-\frac{6}{5} (v^2 + 1)^{-5/6} \right]_a^\infty = \frac{1}{(a^2 + 1)^{-5/6}} = 1 \text{ for } a = 0, = 2^{-5/6} = 0.5612 \text{ for } a = 1. \end{aligned} \quad (15)$$

With the help of Eq. (15), it is easy to estimate D_{vK}^* in Fig. 3 for intermediate values of a .

3. THE SPHERICAL WAVE STRUCTURE FUNCTION

3.1 The Kolmogorov Power Spectral Density of Turbulence

The spherical wave structure function can be found by steps similar to those used for the plane wave structure function. I begin with Eq. (61) from Chap. 6 of Andrews and Phillips [4] and change the variable of integration to $u = K\xi r$:

$$\begin{aligned} D_{sph}(r) &= 8\pi^2 0.033 C_n^2 k^2 Z \int_0^\infty \int_{K_O}^\infty \frac{1 - J_0(K\xi r)}{K^{11/3}} K dK d\xi \\ &= \frac{6(2\pi)^2 0.033 C_n^2 k^2 Z}{5 K_O^{5/3}} \left\{ \frac{5}{3} \int_0^1 (K_O \xi r)^{5/3} \left[\int_0^\infty \frac{1 - J_0(u)}{u^{8/3}} du - \int_0^{K_O \xi r} \frac{1 - J_0(u)}{u^{8/3}} du \right] d\xi \right\}, \end{aligned} \quad (16)$$

from which the normalized structure function is found, as in Eqs. (2) and (4), to be

$$\begin{aligned} D_{sph}^* \left(\frac{r}{L_O} \right) &\approx \int_0^1 \left[1.118 \frac{5}{3} (K_O \xi r)^{5/3} - \frac{5}{4} (K_O \xi r)^2 + \frac{5}{448} (K_O \xi r)^4 \right] d\xi \\ &= 1.118 \frac{5}{8} (K_O r)^{5/3} - \frac{5}{12} (K_O r)^2 + \frac{1}{448} (K_O r)^4 \\ &= 14.95 \left(\frac{r}{L_O} \right)^{5/3} - 16.45 \left(\frac{r}{L_O} \right)^2 + 3.48 \left(\frac{r}{L_O} \right)^4. \end{aligned} \quad (17)$$

D_{sph}^* is evaluated by numerical integration of the double integral in the first line of Eq. (17) and plotted in Figs. 5 and 6, along with the analytic approximations $D_{sph,1,2,3}^*$, which are easily read from the last line of Eq. (17). By repeating the analysis given in Eq. (3), with $v = u/K_O \xi r$, it is easy to see that D_{sph}^* , like D^* , is normalized to an asymptotic value of 1:

$$\begin{aligned} D_{sph}^* \left(\frac{r}{L_O} \right) &\equiv \left[\frac{5}{3} \int_0^1 (K_O \xi r)^{5/3} \int_{K_O \xi r}^\infty \frac{1 - J_0(u)}{u^{8/3}} du d\xi \right] \\ &= \frac{5}{3} \int_0^1 \int_{K_O \xi r}^\infty \frac{1 - J_0(K_O \xi r v)}{v^{8/3}} dv d\xi \rightarrow 1 \text{ as } \frac{r}{L_O} \rightarrow \infty. \end{aligned} \quad (18)$$

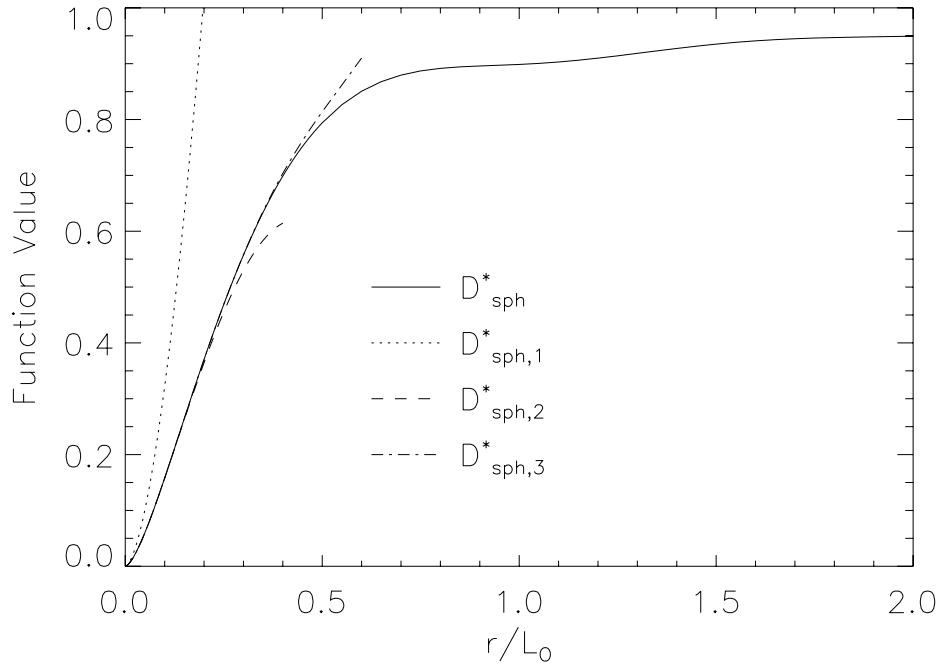


Fig. 5 — Normalized structure functions from Eqs. (16) and (17) for a spherical wave and Kolmogorov spectral density

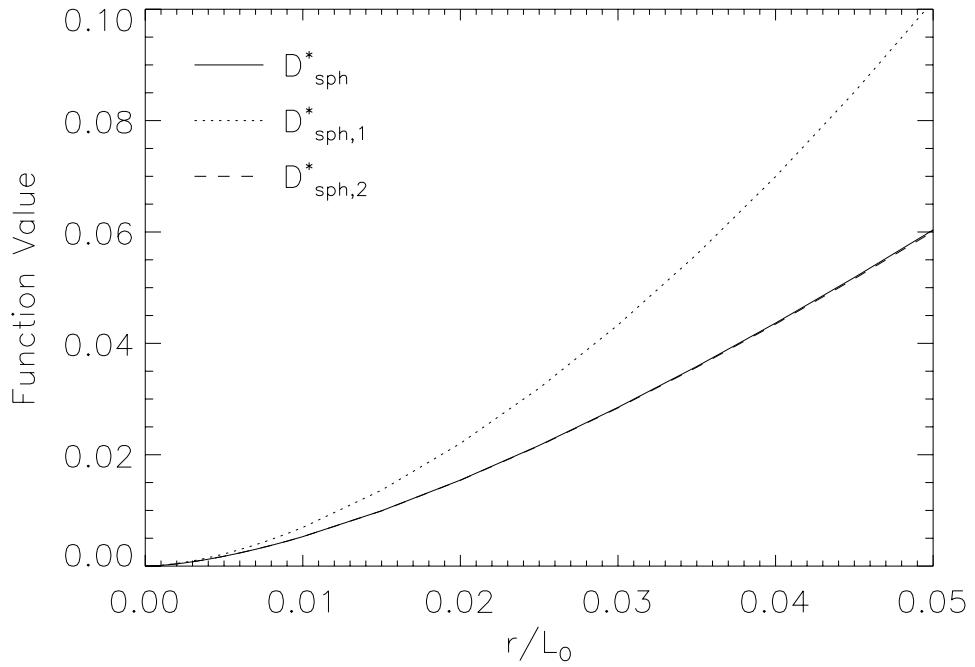


Fig. 6 — Detail of Fig. 5

Substitution of Eq. (17) into Eq. (16) yields

$$\begin{aligned}
 D_{sph}(r) &\approx \frac{6(2\pi)^2 0.033 C_n^2 k^2 Z}{5 K_o^{5/3}} \frac{5}{8} (K_o r)^{5/3} \left[1.118 - \frac{2}{3} (K_o r)^{1/3} + \frac{1}{280} (K_o r)^{7/3} \right] \\
 &= 1.09 C_n^2 k^2 Z r^{5/3} \left[1 - 1.100 \left(\frac{r}{L_o} \right)^{1/3} + 0.233 \left(\frac{r}{L_o} \right)^{7/3} \right] \\
 &= 2.58 \left(\frac{r}{r_0} \right)^{5/3} \left[1 - 1.100 \left(\frac{r}{L_o} \right)^{1/3} + 0.233 \left(\frac{r}{L_o} \right)^{7/3} \right] .
 \end{aligned} \tag{19}$$

The $D_{sph,1,2,3}$ are easily read from Eq. (19).

3.2 The Von Karman Power Spectral Density of Turbulence

I combine the procedures in Sections 3.1 and 2.2. With the von Karman spectral density, Eq. (16) becomes

$$D_{sph,vK}(r) = \frac{6(2\pi)^2 0.033 C_n^2 k^2 Z}{5 K_o^{5/3}} \left[\frac{5}{3} \int_0^1 (K_o \xi r)^{5/3} \int_{a K_o \xi r}^{\infty} \frac{1 - J_0(u)}{(u^2 + K_o^2 \xi^2 r^2)^{11/6}} u du d\xi \right] . \tag{20}$$

The expansion of the second integral in the brackets in Eq. (20) can again be done in terms of hypergeometric functions [8], but it is far easier to repeat the process that led to Eq. (13) and expand this integral in a power series of orders 5/3, 2, and 4 in the quantity $K_o \xi r$. The integral over ξ is then simple, as in Eq. (17). The result is

$$D_{sph,vK,3}^* \left(\frac{r}{L_o} \right) \approx 14.95 \left(\frac{r}{L_o} \right)^{5/3} - 19.7 \left(\frac{r}{L_o} \right)^2 + \begin{cases} 20 & \text{for } a=0 \\ 14 & \text{for } a=1 \end{cases} \left(\frac{r}{L_o} \right)^4 , \tag{21}$$

where the first coefficient ($= 14.95$) comes from Eq. (17), the second ($= -19.7$) from Andrews *et al.*, and the third was found by achieving a visually good fit. These quantities are plotted in Figs. 7 and 8. The von Karman equivalent of Eq. (19) is

$$D_{sph,vK,3}(r) = 2.58 \left(\frac{r}{r_0} \right)^{5/3} \left[1 - 1.32 \left(\frac{r}{L_o} \right)^{1/3} + \begin{cases} 1.34 & \text{for } a=0 \\ 0.94 & \text{for } a=1 \end{cases} \left(\frac{r}{L_o} \right)^{7/3} \right] . \tag{22}$$

Omitting the last term inside the brackets in Eq. (22) yields $D_{sph,vK,2}$, which is the same as Eq. (22) of Andrews *et al.* for $L_I \ll r \ll L_o$.

Equation (18) shows that the asymptotic values of D_{sph}^* and D^* are the same. The same analysis applied to the quantity in brackets in Eq. (20) shows that the asymptotic values of $D_{sph,vK}^*$ and D_{vK}^* are the same, and they are given in Eq. (15).

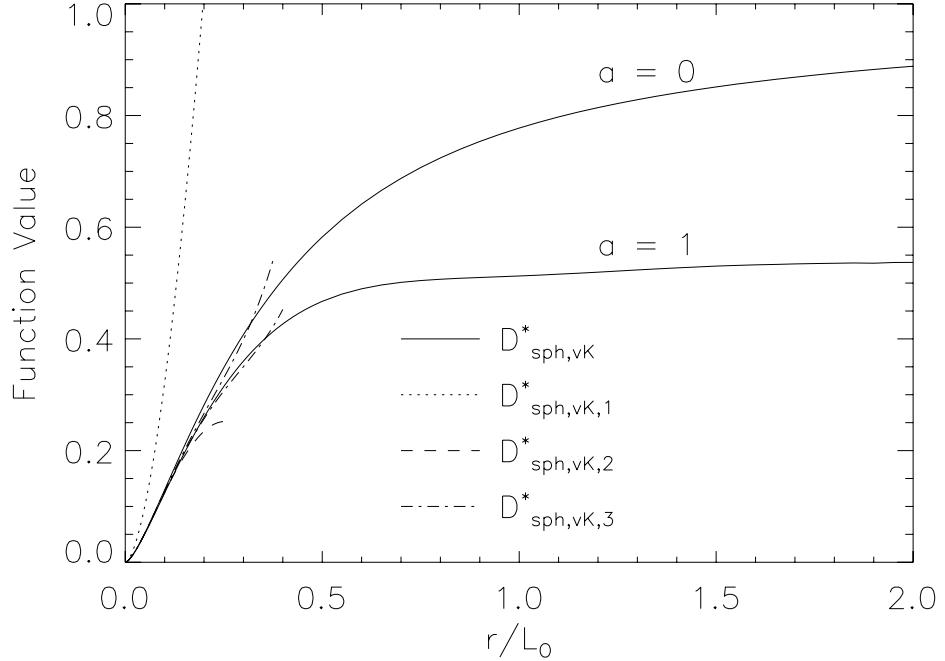


Fig. 7—Similar to Fig. 5, but with von Karman spectral density and functions taken from Eqs. (20) and (21). $D^*_{sph,vK,1} = D^*_{sph,1}$ is the same as in Figs. 5 and 6. Asymptotic values are again given by Eq. (15).

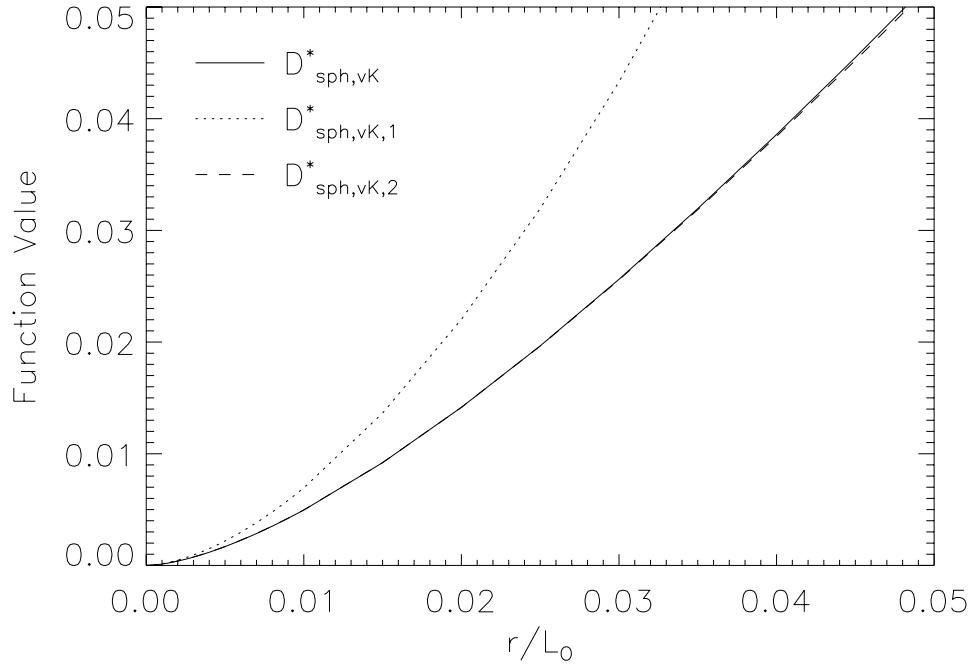


Fig. 8—Detail of Fig. 7

4. THE INDEX-OF-REFRACTION STRUCTURE FUNCTION

I now examine the effect of a finite outer scale on the structure function of the turbulence-induced fluctuations of the atmosphere's index of refraction. While this quantity is not usually of direct interest for the beam-propagation problem, it is part of the derivation of the wave structure function and the reader may be curious. It is normally expressed as $D_n(r) = C_n^2 r^{2/3}$, which is exact only for an infinite outer scale, so the question naturally arises as to how good this approximation is for finite outer scales. I show below that the $r^{2/3}$ approximation to the index structure function, being valid at least up to $r/L_O = 0.1$, is much better than the $r^{5/3}$ approximation is to the wave structure function.

4.1 The Kolmogorov Power Spectral Density of Turbulence

I take the index structure function from Eq. (8.4-21) of Goodman [9], with a lower limit of K_O instead of 0, and apply the usual steps:

$$\begin{aligned} D_n(r) &= 8\pi \times 0.033 C_n^2 \int_{K_O}^{\infty} \frac{1 - S(Kr)}{K^{11/3}} K^2 dK \\ &= \frac{12\pi \times 0.033 C_n^2}{K_O^{2/3}} \left[\frac{2}{3} (K_O r)^{2/3} \int_{K_O r}^{\infty} \frac{1 - S(u)}{u^{5/3}} du \right], \end{aligned} \quad (23)$$

where $S(x) \equiv \sin x / x$. $1 - S(x) \approx x^2/6 - x^4/120$, so the normalized structure function is

$$\begin{aligned} D_n^* \left(\frac{r}{L_O} \right) &\approx \frac{2}{3} (K_O r)^{2/3} \left\{ \int_0^{\infty} \frac{1 - S(u)}{u^{5/3}} du - \int_0^{K_O r} \left(\frac{u^{1/3}}{6} - \frac{u^{7/3}}{120} \right) du \right\} \\ &= \frac{2}{3} (K_O r)^{2/3} \left\{ 1.21 - \frac{(K_O r)^{4/3}}{8} + \frac{(K_O r)^{10/3}}{400} \right\} \\ &= 2.75 \left(\frac{r}{L_O} \right)^{2/3} - 3.29 \left(\frac{r}{L_O} \right)^2 + 2.60 \left(\frac{r}{L_O} \right)^4. \end{aligned} \quad (24)$$

D_n^* , from the brackets in Eq. (23), is evaluated by numerical integration (as is the first integral in Eq. (24)) and plotted in Figs. 9 and 10, along with the $D_n^*_{1,2,3}$. Observe that in this case $D_n^*_{1,2,3}$ is a much better approximation than was the case for the wave structure function: it is good at least to $r/L_O = 0.1$. $D_n^*_{1,2}$ extends the range to 0.3 and $D_n^*_{3,4}$ to 0.5 (being generous). The asymptotic value in Fig. 9 is easily shown to be 1. $D_n(r)$ can now be approximated as

$$\begin{aligned} D_n(r) &\approx \frac{12\pi \times 0.033 C_n^2}{K_O^{2/3}} \frac{2}{3} (K_O r)^{2/3} \left[1.21 - \frac{(K_O r)^{4/3}}{8} + \frac{(K_O r)^{10/3}}{400} \right] \\ &= C_n^2 r^{2/3} \left[1 - 1.20 \left(\frac{r}{L_O} \right)^{4/3} + 0.95 \left(\frac{r}{L_O} \right)^{10/3} \right]. \end{aligned} \quad (25)$$

Equation (25) shows why the correction for finite outer scale is less important for the index structure function than for the wave structure function: the first correction term is proportional to $(r/L_O)^{4/3}$, which is a smaller quantity for small r/L_O than is $(r/L_O)^{1/3}$.

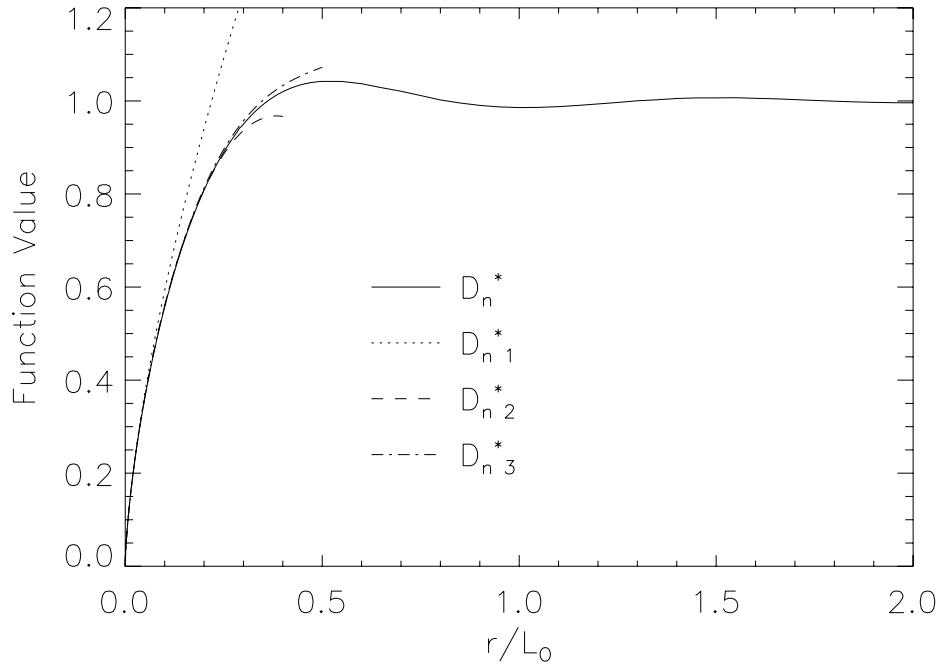


Fig. 9 — Normalized index-of-refraction structure functions for a Kolmogorov spectral density, from Eqs. (23) and (24).

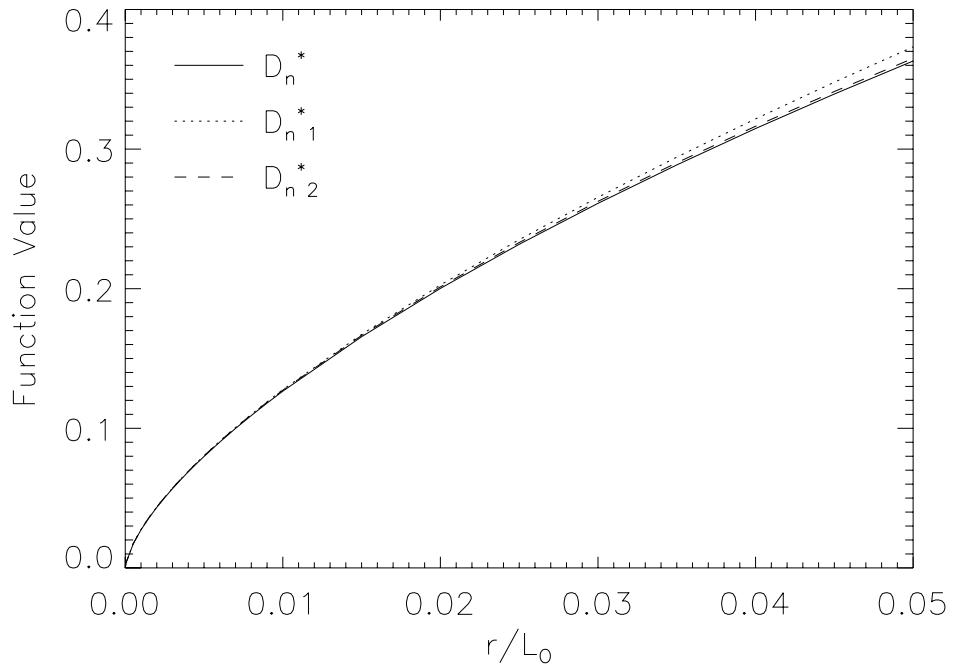


Fig. 10 — Detail of Fig. 9

4.2 The Von Karman Power Spectral Density of Turbulence

For the von Karman spectral density, Eq. (23) becomes

$$D_{n,vK}(r) = \frac{12\pi \times 0.033 C_n^2}{K_o^{2/3}} \left[\frac{2}{3} (K_o r)^{2/3} \int_{aK_o r}^{\infty} \frac{1 - S(u)}{(u^2 + K_o^2 r^2)^{11/6}} u^2 du \right]. \quad (26)$$

In analogy to Section 3.2, but with visual curve fitting used to find both the second and the third coefficients, the normalized structure function is found to be

$$D_{n,vK}^* \left(\frac{r}{L_o} \right) \approx 2.75 \left(\frac{r}{L_o} \right)^{2/3} - 11 \left(\frac{r}{L_o} \right)^2 + 120 \left(\frac{r}{L_o} \right)^4. \quad (27)$$

$D_{n,vK}^*$ is plotted in Figs. 11 and 12 along with the $D_{n,vK,1,2,3}^*$. These figures show that over the range of applicability of Eq. (27), all the coefficients are essentially independent of a . The asymptotic values in Fig. 11 cannot be calculated analytically. Finally,

$$D_{n,vK}(r) \approx C_n^2 r^{2/3} \left[1 - 4.0 \left(\frac{r}{L_o} \right)^{4/3} + 44 \left(\frac{r}{L_o} \right)^{10/3} \right]. \quad (28)$$

5. APPLICATIONS

We have seen that if L_o exceeds the relevant values of r by a factor of at least 100, then the traditional form of the plane wave structure function, $D(r) \approx 6.88(r/r_0)^{5/3}$, is valid. This is often the case for astronomical observations [11-14], but not for horizontal transmission problems near the Earth's surface, where approximations to $D(r)$ that take the outer scale into account are more useful. They are also useful for optical systems with imaging times of a few milliseconds. Over these short times, the atmosphere may be considered fixed, and most problems can identify a scale length, call it L_1 , beyond which disturbances affect only the tilt of a wave front and, therefore, not the resolution of the optical system. Thus, we might select $L_1 = 10D_{ap}$ for an aperture diameter D_{ap} . If (assuming $L_1 \leq L_o$) $\Phi_n(K)$ is cut off at the corresponding wave number K_1 , then the resulting effective structure function $D_{eff}(r)$ is likely to be a better means of evaluating performance than $D(r)$ and, since $D_{ap}/L_1 = 0.1 > 0.01$, is much more accurately given by the forms presented here than by the traditional form, which, as shown in Eqs. (8) and (9), is just a first approximation.

Another application is to computer-generated phase screens that are used to simulate the effect on a propagating wave front of passage through the atmosphere in the presence of turbulence (see, for two examples, Lucke [15] and Lane et al. [16]). One test of whether or not the phase screen has been properly generated is whether or not it generates the correct phase structure function when a plane wave passes through it [16]. After a plane wave has passed through a phase screen, but before it has propagated a significant distance from the screen, its amplitude is unaffected, so the phase structure function and the wave structure function are identical. Therefore, since the width of a phase screen is rarely less than about a meter, while relevant outer scales are often less than 100 m and sometimes less than 10 m, the equations and figures for the various structure functions given here should provide a much better test than Eq. (9) of whether or not a phase screen has been properly generated.

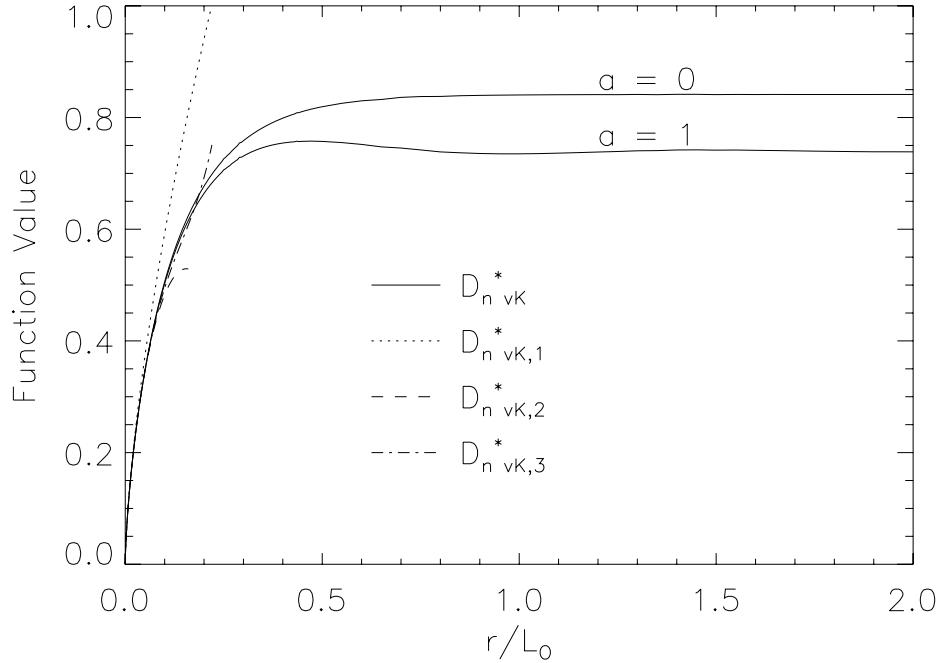


Fig. 11 — Normalized index-of-refraction structure functions for a von Karman spectral density, from Eqs. (26) and (27). $D_n^*_{vK,1} = D_n^*_{vK}$ is the same as in Fig. 9.

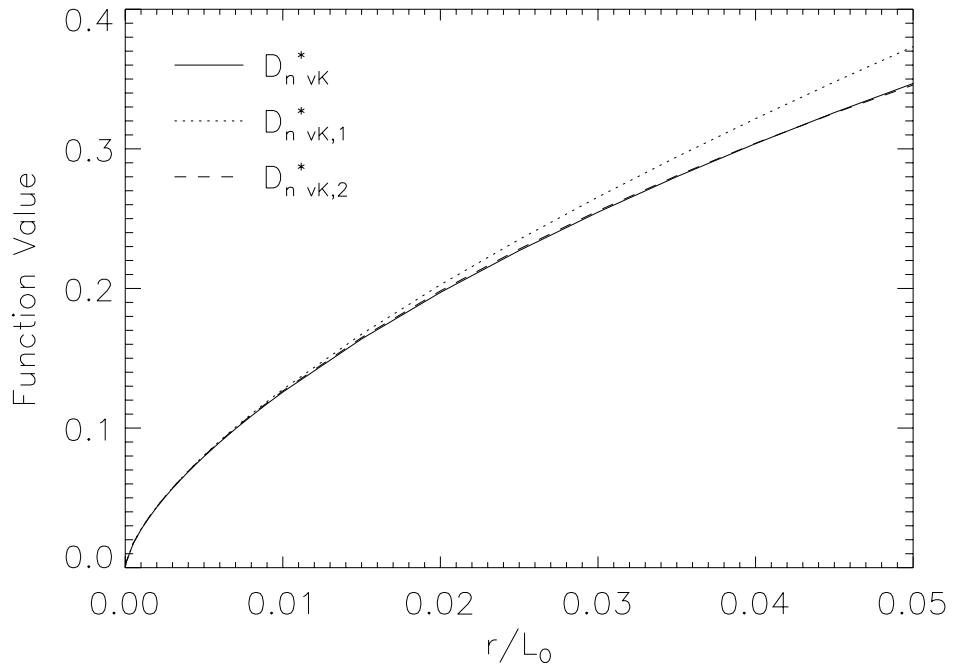


Fig. 12 — Detail of Fig. 11

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Appendix

EVALUATING THE SECOND INTEGRALS IN EQS. (2) AND (12)

The second integrals in Eqs. (2) and (12) are done by using the approximation $1 - J_0(u) \approx u^2/4 - u^4/64$. With this substitution, the second integral in the second line of Eq. (2) becomes

$$\int_0^{K_o r} \frac{u^2/4 - u^4/64}{u^{8/3}} du = \left[\frac{3}{4} u^{1/3} - \frac{3}{7 \times 64} u^{7/3} \right]_{u=0}^{u=K_o r} = \frac{3}{4} (K_o r)^{1/3} - \frac{3}{7 \times 64} (K_o r)^{7/3} . \quad (\text{A1})$$

For the second integral in Eq. (12), we need only the first term in the approximation:

$$\begin{aligned} \int_0^{aK_o r} \frac{u^2/4}{(u^2 + K_o^2 r^2)^{11/6}} u du &= \frac{1}{8} \int_0^{(aK_o r)^2} \frac{u^2}{(u^2 + K_o^2 r^2)^{11/6}} d(u^2) \\ &= \frac{1}{8} \left[\frac{1}{(2-11/6)(u^2 + K_o^2 r^2)^{11/6-2}} + \frac{K_o^2 r^2}{(11/6-1)(u^2 + K_o^2 r^2)^{11/6-1}} \right]_{u^2=0}^{u^2=(aK_o r)^2} \\ &= (K_o r)^{1/3} \frac{1}{8} \left[6(a^2+1)^{1/6} + \frac{6}{5(a^2+1)^{5/6}} - 6 - \frac{6}{5} \right] \\ &= (K_o r)^{1/3} \frac{3}{4} \left[(a^2+1)^{1/6} \left(1 + \frac{1}{5(a^2+1)} \right) - 1.2 \right] \\ &= (K_o r)^{1/3} \times [0 \text{ for } a=0, 0.026 \text{ for } a=1] . \end{aligned} \quad (\text{A2})$$

Finally, $-1.485 - 0.026 = -1.511$.